

A multivariate Gnedenko law of large numbers

Daniel Fresen

In memory of Nigel J. Kalton

ABSTRACT. We show that the convex hull of a large i.i.d. sample from a non-vanishing log-concave distribution approximates a pre-determined body in the logarithmic Hausdorff distance and in the Banach-Mazur distance. For p -log-concave distributions with $1 < p < \infty$ (such as the normal distribution where $p = 2$) we also have approximation in the Hausdorff distance. These are multivariate versions of the Gnedenko law of large numbers which guarantees concentration of the maximum and minimum in the one dimensional case.

We give three different deterministic bodies that serve as approximants to the random body. The first is the floating body that serves as a multivariate quantile, the second body is given as a contour of the density function, and the third body is given in terms of the Radon transform.

We end the paper by constructing a probability measure with an interesting universality property.

1. Introduction

The Gnedenko law of large numbers [7] states that if F is the cumulative distribution of a probability measure μ on \mathbb{R} such that for all $\varepsilon > 0$

$$(1.1) \quad \lim_{t \rightarrow \infty} \frac{F(t + \varepsilon) - F(t)}{1 - F(t + \varepsilon)} = \infty$$

then there are functions δ , T and \mathcal{P} defined on \mathbb{N} with

$$(1.2) \quad \lim_{n \rightarrow \infty} \delta_n = 0$$

$$(1.3) \quad \lim_{n \rightarrow \infty} \mathcal{P}_n = 1$$

such that for any $n \in \mathbb{N}$ and any i.i.d. sample $(\gamma_i)_1^n$ from μ , with probability \mathcal{P}_n we have

$$|\max\{\gamma_i\}_1^n - T_n| < \delta_n$$

The condition (1.1) implies super-exponential decay of the tail probabilities $1 - F(t)$, i.e. for all $c > 0$,

$$\lim_{t \rightarrow \infty} e^{ct}(1 - F(t)) = 0$$

Many thanks to Imre Bárány, John Fresen, Jill Fresen, Nigel Kalton, Alexander Koldobsky, Mathieu Meyer and Mark Rudelson for their comments, advice and support. In particular, I am grateful to Nigel Kalton for his friendship and all that he taught me.

The converse is almost true and can be achieved if we impose some sort of regularity on F . One such regularity condition is log-concavity (see preliminaries). Of course all of this can be re-worded in multiplicative form. Provided $1 - F(t)$ is regular enough and decays super-polynomially, i.e. for any $m \in \mathbb{N}$,

$$\lim_{t \rightarrow \infty} t^m (1 - F(t)) = 0$$

then (1.2) and (1.3) hold, and with probability \mathcal{P}_n ,

$$\left| \frac{\max\{\gamma_i\}_1^n}{T_n} - 1 \right| \leq \delta_n$$

Note that rapid decay of the left hand tail provides concentration of $\min\{\gamma_i\}_1^n$, and that $[\min\{\gamma_i\}_1^n, \max\{\gamma_i\}_1^n] = \text{conv}\{\gamma_i\}_1^n$.

In this paper we extend the Gnedenko law of large numbers to higher dimensions. We consider a collection of i.i.d. random vectors $\{x_i\}_1^n$ in \mathbb{R}^d that follow a log-concave distribution with non-vanishing density and study their convex hull $P_n = \text{conv}\{x_i\}_1^n$. We show that with high probability, P_n approximates a deterministic body.

2. Main Results

Let $d \geq 1$, $n \geq d + 1$ and let μ be a log-concave probability measure on \mathbb{R}^d with non-vanishing density function f . This means that f is of the form $f(x) = \exp(-g(x))$ where g is convex. Let $(x_i)_1^n$ denote a sequence of i.i.d. random vectors in \mathbb{R}^d with distribution μ . The convex body $P_n = \text{conv}\{x_i\}_1^n$ is a random polytope. For any $x \in \mathbb{R}^d$, define

$$\tilde{f}(x) = \inf_{\mathfrak{H}} \mu(\mathfrak{H})$$

where \mathfrak{H} runs through the collection of all half-spaces that contain x . For any $\delta > 0$, we define the *floating body*

$$(2.1) \quad F_\delta = \{x \in \mathbb{R}^d : \tilde{f}(x) \geq \delta\}$$

Note that F_δ is convex and non-empty provided that $\delta < e^{-1}$ (see lemma 5.12 in [11] or lemma 3.3 in [5]). We define the *logarithmic Hausdorff distance* between convex bodies $K, L \subset \mathbb{R}^d$ as,

$$d_{\mathfrak{L}}(K, L) = \inf\{\lambda \geq 1 : \exists x \in \text{int}(K \cap L), \lambda^{-1}(L - x) + x \subset K \subset \lambda(L - x) + x\}$$

The main result of this paper is that for large n the random body P_n approximates the deterministic body $F_{1/n}$ in the logarithmic Hausdorff distance. In particular we prove,

Theorem 1. *For all $q > 0$, all $d \in \mathbb{N}$ and any probability measure μ on \mathbb{R}^d with a non-vanishing log-concave density function, there exist $c, \tilde{c} > 0$ such that for all $n \in \mathbb{N}$ with $n \geq d + 2$, if $(x_i)_1^n$ is an i.i.d. sample from μ , $P_n = \text{conv}\{x_i\}_1^n$ and $F_{1/n}$ is the floating body as in (2.1), then with probability at least $1 - \tilde{c}(\log n)^{-q}$, we have*

$$(2.2) \quad d_{\mathfrak{L}}(P_n, F_{1/n}) \leq 1 + c \frac{\log \log n}{\log n}$$

The strategy of the proof is to use quantitative bounds in the one dimensional case to analyze the Minkowski functional of P_n in different directions. The idea is simple, however there are some subtle complications. The lack of symmetry is a complicating factor, and the fact that the half-spaces of mass $1/n$ do not necessarily touch $F_{1/n}$ adds to the intricacy of the proof.

We define f to be p -log-concave if it is of the form $f(x) = c \exp(-g(x)^p)$ where g is non-negative and convex. For a general log-concave distribution we do not have such an approximation in the Hausdorff distance $d_{\mathcal{H}}$ however we have,

Theorem 2. *For all $q > 0$, $p > 1$ and $d \in \mathbb{N}$, and any probability measure μ on \mathbb{R}^d with a non-vanishing p -log-concave density function, there exist $c, \tilde{c} > 0$ such that for all $n \in \mathbb{N}$ with $n \geq d + 2$, if $(x_i)_1^n$ is an i.i.d. sample from μ , $P_n = \text{conv}\{x_i\}_1^n$ and $F_{1/n}$ is the floating body as in (2.1), then with probability at least $1 - \tilde{c}(\log n)^{-q}$ we have*

$$(2.3) \quad d_{\mathcal{H}}(P_n, F_{1/n}) \leq c \frac{\log \log n}{(\log n)^{1 - \frac{1}{p}}}$$

Theorem 2 can easily be extended to a much larger class of log-concave distributions. Using theorem 1, any bound on the growth rate of $\text{diam}(F_{1/n})$ automatically transfers to a bound on $d_{\mathcal{H}}(P_n, F_{1/n})$. Inequality (2.2) is optimal while inequality (2.3) is optimal for $p = 2$ (see (6.2) and (6.3)).

We also study two other deterministic bodies that serve as approximants to the random body. Define

$$f^{\sharp}(x) = \inf_{\mathcal{H}} \int_{\mathcal{H}} f(y) d_{\mathcal{H}}(y)$$

where \mathcal{H} runs through the collection of all hyperplanes that contain x , and $d_{\mathcal{H}}$ stands for Lebesgue measure on \mathcal{H} . For any $\delta > 0$, define the bodies

$$\begin{aligned} D_{\delta} &= \{x \in \mathbb{R}^d : f(x) \geq \delta\} \\ R_{\delta} &= \{x \in \mathbb{R}^d : f^{\sharp}(x) \geq \delta\} \end{aligned}$$

By log-concavity of f , both D_{δ} and R_{δ} are convex.

Theorem 3. *Let $d \in \mathbb{N}$ and let μ be a probability measure on \mathbb{R}^d with a non-vanishing log-concave density function. Then we have*

$$(2.4) \quad \lim_{\delta \rightarrow 0} d_{\mathcal{L}}(F_{\delta}, D_{\delta}) = 1$$

$$(2.5) \quad \lim_{\delta \rightarrow 0} d_{\mathcal{L}}(F_{\delta}, R_{\delta}) = 1$$

Similar results hold in the Hausdorff distance for log-concave distributions that decay super-exponentially, however we do not discuss this here.

Our prototype example is the class of distributions introduced by Schechtman and Zinn [16] of the form $f(x) = c_p^d \exp(-||x||_p^p)$, where $1 \leq p < \infty$ and $c_p = p/\Gamma(p^{-1})$. In this case $D_{\delta} = (\log(c_p^d/\delta))^{1/p} B_p^d$ and $P_n \approx (\log(c_p^d n))^{1/p} B_p^d$. In fact for these distributions we have a quantitative version of theorem 3 (see remark 1 near the end of section 7). Of particular interest is the Gaussian distribution, where $p = 2$.

It is worth noting that for the standard Gaussian distribution a similar approximation was obtained by Bárány and Vu [4] (see remark 9.6 in their paper) who showed that there exist two radii R and r , both functions of n and d , such that for all $d \geq 2$ both $r, R = (2 \log n)^{1/2}(1 + o(1))$ and with 'high probability'

$rB_2^d \subset P_n \subset RB_2^d$. Their sandwiching result served as a key step in their proof of the central limit theorem for Gaussian polytopes (asymptotic normality of various functionals such as the volume and the number of faces).

The final result of the paper is the following. Let \mathcal{K}_d denote the collection of all convex bodies in \mathbb{R}^d .

Theorem 4. *For all $d \in \mathbb{N}$, there exists a probability measure μ on \mathbb{R}^d with the following universality property. Let $(x_i)_{i=1}^\infty$ be an i.i.d. sample from μ , and for each $n \in \mathbb{N}$ with $n \geq d+1$ let $P_n = \text{conv}\{x_i\}_{i=1}^n$. Then with probability 1, the sequence $(P_n)_{n=d+1}^\infty$ is dense in \mathcal{K}_d with respect to the Banach-Mazur distance.*

Throughout the paper we will make use of variables c , \tilde{c} , c_1 , c_2 , n_0 , m etc. that may depend on other parameters (including dimension) but not on n . Their values may change from one appearance to the next.

3. Preliminaries

Most of the material in this section is discussed in [1], [2], [12] and [13]. We denote the standard Euclidean norm on \mathbb{R}^d by $\|\cdot\|_2$. For any $\varepsilon > 0$, an ε -net in S^{d-1} is a subset \mathcal{N} such that for any distinct $\omega_1, \omega_2 \in \mathcal{N}$, $\|\omega_1 - \omega_2\|_2 > \varepsilon$, and for all $\theta \in S^{d-1}$ there exists $\omega \in \mathcal{N}$ such that $\|\theta - \omega\|_2 \leq \varepsilon$. Such a subset can easily be constructed using induction. By a standard volumetric argument we have

$$(3.1) \quad |\mathcal{N}| \leq \left(\frac{3}{\varepsilon}\right)^d$$

By induction, any $\theta \in S^{d-1}$ can be expressed as a series

$$(3.2) \quad \theta = \omega_0 + \sum_{i=1}^{\infty} \varepsilon_i \omega_i$$

where each $\omega_i \in \mathcal{N}$ and $0 \leq \varepsilon_i \leq \varepsilon^i$. To see this, express $\theta = \omega_0 + r_0$, where $\omega_0 \in \mathcal{N}$ and $\|r_0\|_2 \leq \varepsilon$. Then express $\|r_0\|^{-1}r_0 \in S^{d-1}$ in a similar fashion and iterate this procedure.

Define the functional

$$\|x\|_{\mathcal{N}} = \max\{\langle x, \omega \rangle : \omega \in \mathcal{N}\}$$

As an easy consequence of the Cauchy-Schwarz inequality, provided $\varepsilon \in (0, 1)$ we have

$$(3.3) \quad (1 - \varepsilon)\|x\|_2 \leq \|x\|_{\mathcal{N}} \leq \|x\|_2$$

The geometric meaning of (3.3) is that if we consider the polytope defined by the hyperplanes tangent to B_2^d at points in \mathcal{N} , this body is slightly bigger than B_2^d but not by more than a factor of $(1 - \varepsilon)^{-1}$.

A *convex body* is a compact convex subset of Euclidean space with nonempty interior. For a convex body $K \subset \mathbb{R}^d$ that contains the origin as an interior point, its *Minkowski functional* is defined as

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}$$

for all $x \in \mathbb{R}^d$. By convexity of K , one can easily show that $\|\cdot\|_K$ obeys the triangle inequality. The dual Minkowski functional is defined as

$$\|y\|_{K^\circ} = \sup\{\langle x, y \rangle : x \in K\}$$

for all $y \in \mathbb{R}^d$, and the *polar* of K is

$$K^\circ = \{y \in \mathbb{R}^d : \|y\|_{K^\circ} \leq 1\}$$

By the Hahn-Banach theorem, $K^{\circ\circ} = K$.

The *Banach-Mazur* distance between two convex bodies K and L is defined as

$$d_{BM}(K, L) = \inf\{\lambda \geq 1 : \exists x \in \mathbb{R}^d, \exists T, K \subset TL \subset \lambda(K - x) + x\}$$

where T represents an affine transformation of \mathbb{R}^d . This is a generalization of the classical Banach-Mazur distance between normed spaces (origin symmetric bodies).

The *Hausdorff distance* $d_{\mathcal{H}}$ between K and L is defined as

$$d_{\mathcal{H}}(K, L) = \max\{\max_{k \in K} d(k, L); \max_{l \in L} d(K, l)\}$$

By convexity this reduces to

$$\begin{aligned} d_{\mathcal{H}}(K, L) &= \sup_{\theta \in S^{d-1}} \left| \sup_{k \in K} \langle k, \theta \rangle - \sup_{l \in L} \langle l, \theta \rangle \right| \\ &= \sup_{\theta \in S^{d-1}} |(\|\theta\|_{K^\circ} - \|\theta\|_{L^\circ})| \end{aligned}$$

We define the *logarithmic Hausdorff distance* between K and L about a point $x \in \text{int}(K \cap L)$ as

$$d_{\mathcal{L}}(K, L, x) = \inf\{\lambda \geq 1 : \lambda^{-1}(L - x) + x \subset K \subset \lambda(L - x) + x\}$$

and

$$d_{\mathcal{L}}(K, L) = \inf\{d_{\mathcal{L}}(K, L, x) : x \in \text{int}(K \cap L)\}$$

provided $\text{int}(K \cap L) \neq \emptyset$. Note that

$$\log d_{\mathcal{L}}(K, L, 0) = \sup_{\theta \in S^{d-1}} |\log \|\theta\|_K - \log \|\theta\|_L|$$

The following relations follow directly from the definitions above,

$$\begin{aligned} d_{\mathcal{L}}(K, L, 0) &= d_{\mathcal{L}}(K^\circ, L^\circ, 0) \\ (3.4) \quad d_{\mathcal{L}}(TK, TL) &= d_{\mathcal{L}}(K, L) \end{aligned}$$

where T is any invertible affine transformation. In addition one can easily check that,

$$\begin{aligned} d_{BM}(K, L) &\leq d_{\mathcal{L}}(K, L)^2 \\ (3.5) \quad d_{\mathcal{H}}(K, L) &\leq \text{diam}(K)(d_{\mathcal{L}}(K, L) - 1) \end{aligned}$$

hence all of our bounds in terms of $d_{\mathcal{L}}$ apply equally well to d_{BM} .

By a simple compactness argument, there is an ellipsoid of maximal volume $\mathcal{E}_k \subset K$. This ellipsoid is called the *John ellipsoid* [2] associated to K . It can be shown that \mathcal{E}_k is unique and has the property that $K \subset d(\mathcal{E}_k - x) + x$, where x is the center of \mathcal{E}_k . In particular, $d_{\mathcal{L}}(\mathcal{E}_k, K) \leq d$.

Throughout the paper we will index half-spaces as $\mathfrak{H}_{\theta, t} = \{x \in \mathbb{R}^d : \langle x, \theta \rangle \geq t\}$ and hyperplanes as $\mathcal{H}_{\theta, t} = \{x \in \mathbb{R}^d : \langle x, \theta \rangle = t\}$ where $\theta \in S^{d-1}$ and $t \in \mathbb{R}$. For $\delta > 0$, define the *convex floating body* [17] as $K_\delta = \cap\{\mathfrak{H}_{\theta, t} : \text{vol}_d(K \cap \mathfrak{H}_{\theta, t}) \geq (1 - \delta)\text{vol}_d(K)\}$. Despite the appearance of an inner product in our indexing of half-spaces, the operation $K \mapsto K_\delta$ is independent of Euclidean structure. Note that K_δ is a special case of the body F_δ defined by (2.1) for the case when μ is the uniform probability measure on K . It is known that in this case, the random

polytope P_n is in some sense similar to $K_{1/n}$. As an example, Bárány and Larman [3] proved that

$$c' \text{vol}_d(K \setminus K_{1/n}) \leq \mathbb{E} \text{vol}_d(K \setminus P_n) \leq c'' \text{vol}_d(K \setminus K_{1/n})$$

Of course in this context it is trivial that both in the Hausdorff distance and in the logarithmic Hausdorff distance we have the approximation $P_n \approx K_{1/n}$, as both bodies approximate K . In [6] it is shown that provided $\lambda < 8^{-d}$, we have

$$(3.6) \quad d_{\mathcal{G}}(K, K_\lambda, x) \leq 1 + 8\lambda^{1/d}$$

where x is the centroid of K . The *cone measure* on ∂K is defined as

$$\mu_K(E) = \text{vol}_d(\{r\theta : \theta \in E, r \in [0, 1]\})$$

for all measurable $E \subset \partial K$. The significance of the cone measure is that it leads to a natural polar integration formula (see [14]); for all $f \in L_1(\mathbb{R}^d)$,

$$(3.7) \quad \int_{\mathbb{R}^d} f(x) dx = d \int_0^\infty \int_{\partial K} r^{d-1} f(r\theta) d\mu_K(\theta) dr$$

A function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is called *log-concave* (see [9]) if

$$f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^d$ and all $\lambda \in (0, 1)$. As a consequence of the Prékopa-Leindler inequality [1], if x is a random vector with log-concave density and y is any fixed vector, then $\langle x, y \rangle$ has a log-concave density in \mathbb{R} . In this paper we will consider probability measures μ with non-vanishing log-concave densities. Clearly, such density functions can be written in the form

$$f(x) = e^{-g(x)}$$

where $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is convex and $\lim_{x \rightarrow \infty} g(x) = \infty$. Any such g lies above a cone, i.e.

$$(3.8) \quad g(x) \geq m\|x\|_2 - c$$

with $m, c > 0$. Hence a log-concave function such as f must decay exponentially to zero (with *uniform* exponential decay rate in all directions). Log-concave functions are very rigid. One such example of this rigidity (see lemma 5.12 in [11]) is the fact that if \mathfrak{H} is any half-space containing the centroid of μ , then $\mu(\mathfrak{H}) \geq e^{-1}$.

Let $1 \leq p < \infty$. If $g : \mathbb{R}^d \rightarrow [0, \infty)$ is convex and $\lim_{x \rightarrow \infty} g(x) = \infty$, then the probability distribution with density given by

$$f(x) = ce^{-g(x)^p}$$

will be called *p-log-concave*. This is a natural generalization of the normal distribution. The 1-log-concave distributions are precisely the non-vanishing log-concave distributions, and if f is *p-log-concave*, then it is also *p'-log-concave* for all $1 \leq p' \leq p$.

Let \mathbf{H}_d denote the collection of all $d - 1$ dimensional affine subspaces (hyperplanes) of \mathbb{R}^d . The Radon transform of a log-concave function $f : \mathbb{R}^d \rightarrow [0, \infty)$ is the function $Rf : \mathbf{H}_d \rightarrow [0, \infty)$ defined by

$$Rf(\mathcal{H}) = \int_{\mathcal{H}} f(y) d_{\mathcal{H}}(y)$$

where $d_{\mathcal{H}}$ is Lebesgue measure on \mathcal{H} . The Radon transform is closely related to the Fourier transform. See [10] for a discussion of these operators and their connections to convex geometry

4. The one dimensional case

Let f be a non-vanishing log-concave probability density function on \mathbb{R} associated to a probability measure μ . In particular, $f(t) = e^{-g(t)}$ where $g : \mathbb{R} \rightarrow \mathbb{R}$ is convex. For $t \in \mathbb{R}$, define

$$\begin{aligned} J(t) &= \int_{-\infty}^t f(s) ds \\ u(t) &= -\log(1 - J(t)) \end{aligned}$$

The cumulative distribution function J is a strictly increasing bijection between \mathbb{R} and $(0, 1)$. The following lemma is a standard result (see e.g. theorem 5.1 in [11] for the statement, and the references given there). However we include a short proof here for completeness.

Lemma 1. *u is convex*

PROOF. Assume momentarily that $g \in C^2(\mathbb{R})$. For $t \in (0, 1)$ define

$$\psi(t) = f(J^{-1}(1 - t))$$

Note that

$$\psi''(t) = \frac{-g''(J^{-1}(1 - t))}{\psi(t)} \leq 0$$

Hence ψ is concave. In addition, $\lim_{t \rightarrow 0} \psi(t) = \lim_{t \rightarrow 1} \psi(t) = 0$. Hence, the function $\kappa(t) = \psi(t)/t$ is non-increasing on $(0, 1)$ and the function $f(t)/(1 - J(t)) = \kappa(1 - J(t))$ is non-decreasing on \mathbb{R} . Since $u'(t) = f(t)/(1 - J(t))$, u is convex.

If $g \notin C^2(\mathbb{R})$, then the result follows by approximation (convolve μ with a Gaussian). \square

The following lemma is a quantitative version of the Gnedenko law of large numbers for log-concave probability measures on \mathbb{R} .

Lemma 2. *For all $q > 0$ there exist $c, \tilde{c} > 0$ such that for all $n \in \mathbb{N}$ with $n \geq 3$, if μ is a probability measure on \mathbb{R} with a non-vanishing log-concave density function and cumulative distribution function J , and $(\gamma_i)_1^n$ is an i.i.d. sample from μ , then with probability at least $1 - \tilde{c}(\log n)^{-q}$ we have*

$$(4.1) \quad \frac{|\gamma_{(n)} - J^{-1}(1 - 1/n)|}{J^{-1}(1 - 1/n) - \mathbb{E}\mu} \leq c \frac{\log \log n}{\log n}$$

where $\gamma_{(n)} = \max\{\gamma_i\}_1^n$ and $\mathbb{E}\mu$ denotes the mean of μ .

PROOF. Let $a = (\log n)^{-q}$ and $b = q \log n$. By choosing an appropriate value of \tilde{c} , the probability bound becomes trivial for all $n \leq n_0$; we may therefore assume that $n > n_0$. As mentioned in the preliminaries (see also lemma 3.3 in [5]), $1 - J(\mathbb{E}\mu) \geq e^{-1}$, hence $u(\mathbb{E}\mu) \leq 1$. Note that by convexity of u we have the inequality $(s - \mathbb{E}\mu)^{-1}(u(s) - u(\mathbb{E}\mu)) \leq (t - s)^{-1}(u(t) - u(s))$ which is valid provided that $\mathbb{E}\mu < s < t$. By setting $s = J^{-1}(1 - b/n)$ and $t = J^{-1}(1 - a/n)$ this inequality becomes

$$(4.2) \quad \frac{J^{-1}(1 - a/n) - J^{-1}(1 - b/n)}{J^{-1}(1 - b/n) - \mathbb{E}\mu} \leq \frac{\log b - \log a}{\log n - \log b - 1}$$

By independence,

$$\begin{aligned} \mathbb{P}\{J^{-1}(1-b/n) \leq \gamma_{(n)} \leq J^{-1}(1-a/n)\} \\ &= \left(1 - \frac{a}{n}\right)^n - \left(1 - \frac{b}{n}\right)^n \\ &\geq 1 - a - e^{-b} \end{aligned}$$

and if the event $\{J^{-1}(1-b/n) \leq \gamma_{(n)} \leq J^{-1}(1-a/n)\}$ occurs, then the conclusion of the theorem holds. \square

5. The multi-dimensional case

Lemma 3. *Let $d \in \mathbb{N}$, $d \geq 1$ and let K and L be convex bodies in \mathbb{R}^d such that $rB_2^d \subset K \subset RB_2^d$ for some $r, R > 0$. Let $0 < \rho < 1/2$ and $0 < \varepsilon < (16R/r)^{-1}$, and let \mathcal{N} be an ε -net in S^{d-1} . Suppose that for each $\omega \in \mathcal{N}$,*

$$(5.1) \quad (1 - \rho)\|\omega\|_L \leq \|\omega\|_K \leq (1 + \rho)\|\omega\|_L$$

Then for all $x \in \mathbb{R}^d$ we have

$$(5.2) \quad (1 + 2\rho + 28Rr^{-1}\varepsilon)^{-1}\|x\|_L \leq \|x\|_K \leq (1 + 2\rho + 28Rr^{-1}\varepsilon)\|x\|_L$$

In particular,

$$(5.3) \quad d_{\mathcal{E}}(K, L) \leq d_{\mathcal{E}}(K, L, 0) \leq 1 + 2\rho + 28Rr^{-1}\varepsilon$$

PROOF. Note that $1 + \rho \leq (1 - \rho)^{-1} \leq 1 + 2\rho$ and $1 - \rho \leq (1 + \rho)^{-1} \leq 1 - \rho/2$, and the same inequalities hold for ε . Since $rB_2^d \subset K \subset RB_2^d$, we have that

$$R^{-1}\|x\|_2 \leq \|x\|_K \leq r^{-1}\|x\|_2$$

for all $x \in \mathbb{R}^d$. Combining this with (5.1) gives

$$R^{-1}(1 + \rho)^{-1} \leq \|\omega\|_L \leq r^{-1}(1 - \rho)^{-1}$$

for all $\omega \in \mathcal{N}$. Consider any $\theta \in S^{d-1}$. By the series representation (3.2) and the triangle inequality,

$$\|\theta\|_L \leq r^{-1}(1 - \rho)^{-1}(1 - \varepsilon)^{-1}$$

Hence $\|x\|_L \leq r^{-1}(1 - \rho)^{-1}(1 - \varepsilon)^{-1}\|x\|_2$ for all $x \in \mathbb{R}^d$. Using the triangle inequality in a bit of a different way,

$$\begin{aligned} \|\theta\|_L &\geq \|\omega_0\|_L - \sum_{i=1}^{\infty} \varepsilon_i \|\omega_i\|_L \\ &\geq R^{-1}(1 + \rho)^{-1} - r^{-1}\varepsilon(1 - \varepsilon)^{-1}(1 - \rho)^{-1} \\ &\geq R^{-1}/2 - 4r^{-1}\varepsilon \\ &= R^{-1}(1 - 8Rr^{-1}\varepsilon)/2 \\ &\geq (4R)^{-1} \end{aligned}$$

which holds since $8Rr^{-1}\varepsilon \leq 1/2$. Thus,

$$\begin{aligned}
\|\theta\|_L &\leq \|\omega_0\|_L + \|\theta - \omega_0\|_L \\
&\leq (1 - \rho)^{-1}\|\omega_0\|_K + r^{-1}(1 - \rho)^{-1}(1 - \varepsilon)^{-1}\varepsilon \\
&\leq (1 - \rho)^{-1}(\|\theta\|_K + \|\omega_0 - \theta\|_K) + r^{-1}(1 - \rho)^{-1}(1 - \varepsilon)^{-1}\varepsilon \\
&\leq (1 - \rho)^{-1}\|\theta\|_K + r^{-1}(1 - \rho)^{-1}\varepsilon(1 + (1 - \varepsilon)^{-1}) \\
&\leq (1 - \rho)^{-1}\|\theta\|_K + Rr^{-1}(1 - \rho)^{-1}\varepsilon(1 + (1 - \varepsilon)^{-1})\|\theta\|_K \\
&\leq (1 + 2\rho)(1 + 3Rr^{-1}\varepsilon)\|\theta\|_K \\
&\leq (1 + 2\rho + 6Rr^{-1}\varepsilon)\|\theta\|_K
\end{aligned}$$

where ω_0 is the element of \mathcal{N} that minimizes $\|\theta - \omega_0\|_2$. On the other hand,

$$\begin{aligned}
\|\theta\|_K &\leq \|\omega_0\|_K + \|\theta - \omega_0\|_K \\
&\leq (1 + \rho)\|\omega_0\|_L + r^{-1}\varepsilon \\
&\leq (1 + \rho)(\|\theta\|_L + \|\omega_0 - \theta\|_L) + r^{-1}\varepsilon \\
&\leq (1 + \rho)\|\theta\|_L + r^{-1}(1 + \rho)(1 - \rho)^{-1}(1 - \varepsilon)^{-1}\varepsilon + r^{-1}\varepsilon \\
&\leq (1 + \rho)\|\theta\|_L + 7r^{-1}\varepsilon \cdot 4R\|\theta\|_L \\
&\leq (1 + \rho + 28Rr^{-1}\varepsilon)\|\theta\|_L
\end{aligned}$$

The result follows by positive homogeneity. \square

PROOF OF THEOREM 1. By choosing \tilde{c} large enough, the probability bound becomes trivial for $n < n_0$. Let $n \geq n_0$ and $\varepsilon = (\log n)^{-1}$. By applying a suitable affine transformation, we may assume that the John ellipsoid of $D_{1/n}$ is ηB_2^d for some $\eta > 0$. We need to use an affine transformation of the form $Tx = Mx + x_0$ where $|\det M| = 1$. Such a transformation preserves Lebesgue measure and therefore preserves the relationship between μ and $D_{1/n}$. Thus $\eta B_2^d \subset D_{1/n} \subset d\eta B_2^d$. If $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle x, \theta \rangle \geq t\}$ is a half-space with $\mu(\mathfrak{H}_{\theta,t}) = 1/n$, then $\eta/2 \leq t \leq 2d\eta$ (this is nothing but a very coarse version of the results of section 7). Hence $\eta/2B_2^d \subset F_{1/n} \subset 2d\eta B_2^d$, and ηB_2^d acts almost like a John ellipsoid for $F_{1/n}$.

For each $\theta \in S^{d-1}$, the function $f_\theta(t) = -\frac{d}{dt}\mu(\mathfrak{H}_{\theta,t})$ is the density of a log-concave probability measure μ_θ on \mathbb{R} with cumulative distribution function $J_\theta(t) = 1 - \mu(\mathfrak{H}_{\theta,t})$. Furthermore, the sequence $(\langle \theta, x_i \rangle)_{i=1}^n$ is an i.i.d. sample from this distribution. Recalling the definition of the dual Minkowski functional, for any $y \in \mathbb{R}^d$

$$\begin{aligned}
\|y\|_{P_n^\circ} &= \sup\{\langle x, y \rangle : x \in P_n\} \\
&= \max_{i=1 \dots n} \langle x_i, y \rangle
\end{aligned}$$

We use this notation even when $0 \notin P_n$. Let \mathcal{N} denote a generic ε -net in S^{d-1} and consider the function

$$\tilde{f}_{\mathcal{N}}(x) = \inf\{\mu(\mathfrak{H}_{\omega,t}) : \omega \in \mathcal{N}, t = \langle \omega, x \rangle\}$$

For all $\delta > 0$, define the 'floating polytope'

$$F_\delta^{\mathcal{N}} = \{x \in \mathbb{R}^d : \tilde{f}_{\mathcal{N}}(x) \geq \delta\}$$

Note that $\tilde{f}(x) = \inf_{\mathcal{N}} \tilde{f}_{\mathcal{N}}(x)$ and $F_\delta = \cap_{\mathcal{N}} F_\delta^{\mathcal{N}}$, where \mathcal{N} runs through the collection of all ε -nets in S^{d-1} . By the geometric interpretation of (3.3) we have

$\eta/2B_2^d \subset F_{1/n}^\mathcal{N} \subset 3d\eta B_2^d$ and therefore $(3d\eta)^{-1}B_2^d \subset (F_{1/n}^\mathcal{N})^\circ \subset 2\eta^{-1}B_2^d$. For each $\theta \in S^{d-1}$, we have

$$\begin{aligned}\mathbb{E}\mu_\theta &\geq J_\theta^{-1}(e^{-1}) \\ &\geq J_\theta^{-1}(1/n)\end{aligned}$$

Combining this and (4.1), with probability at least $1 - \tilde{c}(\log n)^{-d-q}$ we have that,

$$\frac{||\theta||_{P_n^\circ} - J_\theta^{-1}(1-1/n)}{J_\theta^{-1}(1-1/n) - J_\theta^{-1}(1/n)} \leq c \frac{\log \log n}{\log n}$$

Since both $-J_\theta^{-1}(1/n)$ and $J_\theta^{-1}(1-1/n)$ lie in the interval $[\eta/2, 2d\eta]$, both have roughly the same order of magnitude and we have

$$(1-\rho)J_\theta^{-1}(1-1/n) \leq ||\theta||_{P_n^\circ} \leq (1+\rho)J_\theta^{-1}(1-1/n)$$

where

$$\rho = c \frac{\log \log n}{\log n}$$

With probability at least $1 - \tilde{c}\varepsilon^{-d}(\log n)^{-d-q} = 1 - \tilde{c}(\log n)^{-q}$, this happens for all $\omega \in \mathcal{N}$. Hence,

$$(1-\rho)P_n \subset F_{1/n}^\mathcal{N}$$

which implies that

$$(1-\rho)||\theta||_{P_n^\circ} \leq ||\theta||_{(F_{1/n}^\mathcal{N})^\circ}$$

for all $\theta \in S^{d-1}$. On the other hand, for all $\omega \in \mathcal{N}$ we have

$$\begin{aligned}||\omega||_{P_n^\circ} &\geq (1-\rho)J_\omega^{-1}(1-1/n) \\ &\geq (1-\rho)||\omega||_{(F_{1/n}^\mathcal{N})^\circ}\end{aligned}$$

By (5.2),

$$(5.4) \quad (1+2\rho+168d\varepsilon)^{-1}||x||_{P_n^\circ} \leq ||x||_{(F_{1/n}^\mathcal{N})^\circ} \leq (1+2\rho+168d\varepsilon)||x||_{P_n^\circ}$$

for all $x \in \mathbb{R}^d$. Let \mathcal{M} be any other ε -net in S^{d-1} . By the calculations above, with probability at least $1 - \tilde{c}(\log n)^{-q}$,

$$(5.5) \quad (1+2\rho+168d\varepsilon)^{-1}||x||_{P_n^\circ} \leq ||x||_{(F_{1/n}^\mathcal{M})^\circ} \leq (1+2\rho+168d\varepsilon)||x||_{P_n^\circ}$$

for all $x \in \mathbb{R}^d$. By the union bound, with probability at least $1 - \tilde{c}(\log n)^{-q} > 0$, both (5.4) and (5.5) hold. Since both $F_{1/n}^\mathcal{N}$ and $F_{1/n}^\mathcal{M}$ are deterministic bodies, the only way that this can be true is if

$$(1+2\rho+168d\varepsilon)^{-2}F_{1/n}^\mathcal{N} \subset F_{1/n}^\mathcal{M} \subset (1+2\rho+168d\varepsilon)^2F_{1/n}^\mathcal{N}$$

Since $F_{1/n} = \cap_{\mathcal{M}} F_{1/n}^\mathcal{M}$, where the intersection is taken over all ε -nets in S^{d-1} , we have

$$(1+2\rho+168d\varepsilon)^{-2}F_{1/n}^\mathcal{N} \subset F_{1/n} \subset (1+2\rho+168d\varepsilon)^2F_{1/n}^\mathcal{N}$$

Combining this with the polar of (5.4) gives that with probability at least $1 - \tilde{c}(\log n)^{-q}$ we have

$$(1+2\rho+168d\varepsilon)^{-3}P_n \subset F_{1/n} \subset (1+2\rho+168d\varepsilon)^3P_n$$

from which the result follows. \square

Lemma 4. *Let $g : \mathbb{R}^d \rightarrow [0, \infty)$ be convex with $\lim_{x \rightarrow \infty} g(x) = \infty$, let $K \subset \mathbb{R}^d$ be a convex body containing 0 in its interior, and let $p > 1$. Then there exist $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^d$,*

$$(5.6) \quad g(x)^p \geq c_1 \|x\|_K^p - c_2$$

PROOF. We leave the easy proof of this to the reader. \square

Lemma 5. *Let $p > 1$, $d \in \mathbb{N}$ and let μ be a p -log-concave probability measure on \mathbb{R}^d . Then there exist $c_1, c_2, t_0 > 0$ such that for all $\theta \in S^{d-1}$ and all $t \geq t_0$,*

$$(5.7) \quad \mu(\mathfrak{H}_{\theta,t}) \leq c_1 t^{1-p} e^{-c_2 t^p}$$

where $\mathfrak{H}_{\theta,t} = \{x \in \mathbb{R}^d : \langle x, \theta \rangle \geq t\}$.

PROOF. For all $t \geq 1$ we have

$$\begin{aligned} e^{-t^p} &\leq -\frac{d}{dt} \left(p^{-1} t^{1-p} e^{-t^p} \right) \\ &= p^{-1} (p-1) t^{-p} e^{-t^p} + e^{-t^p} \\ &\leq p^{-1} (2p-1) e^{-t^p} \end{aligned}$$

Hence, by the fundamental theorem of calculus,

$$(5.8) \quad (2p-1)^{-1} t^{1-p} e^{-t^p} \leq \int_t^\infty e^{-s^p} ds \leq p^{-1} t^{1-p} e^{-t^p}$$

Since the image of a p -log-concave probability measure under an orthogonal transformation is p -log-concave, we may assume without loss of generality that $\theta = e_1 = (1, 0, 0, \dots)$. By (5.6), there exist $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^d$,

$$f(x) \leq c_1 e^{-c_2 \|x\|_p^p}$$

where $\|x\|_p^p = \sum_{i=1}^d |x_i|^p$. Hence,

$$\begin{aligned} \mu(\mathfrak{H}_{\theta,t}) &\leq \int_{\mathfrak{H}_{\theta,t}} c_1 e^{-c_2 \|x\|_p^p} dx \\ (5.9) \quad &= \int_t^\infty c_3 e^{-c_2 s^p} ds \end{aligned}$$

The result now follows from a change of variables, (5.9) and (5.8). \square

PROOF OF THEOREM 2: By (5.7) $\text{diam}(F_{1/n}) \leq c(\log n)^{1/p}$. The result now follows from (2.2) and (3.5). \square

6. Optimality

Let Φ denote the cumulative standard normal distribution on \mathbb{R} ,

$$\Phi(t) = (2\pi)^{-1/2} \int_{-\infty}^t e^{-\frac{1}{2}s^2} ds$$

By (5.8) there exists $c > 0$ such that for all $n \geq 3$,

$$(6.1) \quad \Phi^{-1}(1 - 1/n) \geq c(\log n)^{1/2}$$

Lemma 6. *For all $q > 0$ and all $d \in \mathbb{N}$, there exists $c, \tilde{c} > 0$ such that for all $n \geq d + 1$, if $(x_i)_1^n$ is an i.i.d. sample from the standard normal distribution on \mathbb{R}^d and $P_n = \text{conv}\{x_i\}_1^n$, then with probability at least $1 - \tilde{c}(\log n)^{-q(d-1)/2}$ both of the following events occur,*

$$(6.2) \quad d_{\mathcal{H}}(P_n, F_{1/n}) \geq c(\log n)^{-\frac{1}{2}-q}$$

$$(6.3) \quad d_{\mathcal{L}}(P_n, F_{1/n}) \geq 1 + c(\log n)^{-1-q}$$

PROOF. A standard result in approximation theory ([8] p. 326) is that for any polytope $K_m \subset \mathbb{R}^d$ with at most m vertices,

$$(6.4) \quad d_{\mathcal{H}}(K_m, B_2^d) > c \left(\frac{1}{m} \right)^{\frac{2}{d-1}}$$

Since $F_{1/n} = \Phi^{-1}(1 - 1/n)B_2^d$, inequality (6.1) implies that

$$d_{\mathcal{H}}(K_m, F_{1/n}) > c(\log n)^{1/2} \left(\frac{1}{m} \right)^{\frac{2}{d-1}}$$

By a result of Raynaud [15], the number of vertices of P_n , denoted by $f_0(P_n)$, obeys the inequality $\mathbb{E}f_0(P_n) < \tilde{c}(\log n)^{(d-1)/2}$. By Chebychev's inequality we have

$$\mathbb{P}\{f_0(P_n) > (\log n)^{\frac{(d-1)(q+1)}{2}}\} \leq \frac{\mathbb{E}f_0(P_n)}{(\log n)^{\frac{(d-1)(q+1)}{2}}} < \tilde{c}(\log n)^{-\frac{q(d-1)}{2}}$$

and if this event occurs then so does (6.2). By (6.4) and (3.5) we get

$$d_{\mathcal{L}}(K_m, B_2^d) > 1 + c \left(\frac{1}{m} \right)^{\frac{2}{d-1}}$$

Since $d_{\mathcal{L}}$ is preserved by invertible affine transformations (as per (3.4)), the same inequality holds for all Euclidean balls. This gives (6.3). \square

We can choose q to be arbitrarily small, in which case (6.2) and (6.3) complement (2.3) and (2.2).

7. Proof of theorem 3

Let f be the density of μ and let $g(x) = -\log f(x)$. Consider the function $\zeta_{\mu} : (0, \infty) \rightarrow (0, 1]$ defined by

$$\zeta_{\mu}(\varepsilon) = \mu(\mathbb{R}^d \setminus D_{\varepsilon})$$

Lemma 7. *There exist $c, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$(7.1) \quad \zeta_{\mu}(\varepsilon) \leq c\varepsilon (\log \varepsilon^{-1})^d$$

PROOF. By inequality (3.8), there exist $m, \tilde{c} > 0$ such that for all $x \in \mathbb{R}^d$,

$$f(x) \leq e^{-m\|x\|_2 + \tilde{c}}$$

Hence there exists $c_1 > 0$ such that for all $\varepsilon \in (0, 1/2)$,

$$\text{vol}_d(D_{\varepsilon}) \leq c_1 (\log \varepsilon^{-1})^d$$

Again by (3.8), there exists $R > 0$ such that $\inf\{g(x) : \|x\|_2 = R\} > g(0)$. Let

$$\alpha = \frac{\inf\{g(x) : \|x\|_2 = R\} - g(0)}{R}$$

For any $\varepsilon < \inf\{f(x) : \|x\| \leq R\}$ and $\theta \in \partial D_\varepsilon$, $\|\theta\|_2 > R$. By convexity of g , for any $r \geq 1$ we have $g(r\theta) \geq g(\theta) + \alpha(r-1)\|\theta\|_2 \geq g(\theta) + \alpha R(r-1)$. By using the polar integration formula (3.7) for D_ε we arrive at

$$\begin{aligned} \zeta_\mu(\varepsilon) &= \int_{\mathbb{R}^d \setminus D_\varepsilon} f(x) dx \\ &= d \int_1^\infty \int_{\partial D_\varepsilon} r^{d-1} f(r\theta) d\mu_{D_\varepsilon}(\theta) dr \\ &\leq d \int_1^\infty \int_{\partial D_\varepsilon} r^{d-1} \varepsilon e^{-\alpha R(r-1)} d\mu_{D_\varepsilon}(\theta) dr \\ &= c_2 \varepsilon \text{vol}_d(D_\varepsilon) \\ &\leq c \varepsilon (\log \varepsilon^{-1})^d \end{aligned}$$

where $c_2 = d \int_1^\infty r^{d-1} e^{-\alpha R(r-1)} dr$, $c = c_1 \cdot c_2$, and we have used the identity $\mu_{D_\varepsilon}(\partial D_\varepsilon) = \text{vol}_d(D_\varepsilon)$ which follows from the definition of μ_{D_ε} . \square

Lemma 8. *For any $x \in \mathbb{R}^d$ there exist $c', \delta_0 > 0$ and a function $p : (0, \delta_0) \rightarrow (0, \infty)$ such that for all $\delta \in (0, \delta_0)$,*

$$(7.2) \quad p(\delta) \leq c' \frac{\log \log \delta^{-1}}{\log \delta^{-1}}$$

and

$$(7.3) \quad F_\delta \subset (1+p)(D_\delta - x) + x$$

PROOF. Let $c > 0$ be the constant in (7.1). A brief analysis of the function $t \mapsto ct(\log t^{-1})^d$ shows that there exists $\delta_0 > 0$ and a function $\varepsilon = \varepsilon(\delta)$ defined implicitly for all $\delta \in (0, \delta_0)$ by the equation $\delta = c\varepsilon(\log \varepsilon^{-1})^d$. We can take δ_0 small enough to ensure that $\varepsilon < \delta$ and that $\log \delta^{-1} < \log \varepsilon^{-1} < 2 \log \delta^{-1}$. If we define

$$p(\delta) = 3 \frac{\log \varepsilon^{-1} - \log \delta^{-1}}{\log \delta^{-1}}$$

then $\delta^{1+p/2} < \varepsilon$ and (7.2) holds. By (7.1), $F_\delta \subset D_\varepsilon$. For any $\theta \in S^{d-1}$ consider the function $f_\theta(t) = f(x + t\theta) = e^{-g_\theta(t)}$, $t \in \mathbb{R}$. This notation differs slightly from that in the proof of theorem 1. If ε is small enough then for all $\theta \in S^{d-1}$ there is a unique $v > 0$ such that $f_\theta(v) = \varepsilon$; we denote this number by $f_\theta^{-1}(\varepsilon)$. We may assume that $\delta_0 < \min\{1, f(x)^2\}$. Note that $1 < \delta/\varepsilon < \delta^{-p/2}$ and $\log \delta^{-1} + \log f(x) \geq 1/2 \log \delta^{-1}$. By convexity of g_θ , for any $0 < s < v$ we have $s^{-1}(g_\theta(s) - g_\theta(0)) \leq (v-s)^{-1}(g_\theta(v) - g_\theta(s))$. Taking $v = f_\theta^{-1}(\varepsilon)$ and $s = f_\theta^{-1}(\delta)$, this becomes

$$(7.4) \quad \frac{f_\theta^{-1}(\varepsilon) - f_\theta^{-1}(\delta)}{f_\theta^{-1}(\delta)} \leq \frac{\log \varepsilon^{-1} - \log \delta^{-1}}{\log \delta^{-1} + \log f(x)} < p$$

Inequality (7.4) reduces to $f_\theta^{-1}(\varepsilon) \leq (1+p)f_\theta^{-1}(\delta)$. Since this holds for any $\theta \in S^{d-1}$, $D_\varepsilon \subset (1+p)(D_\delta - x) + x$ and (7.3) follows. \square

Lemma 9. *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ we have the relation*

$$(7.5) \quad (1 + 8\lambda^{1/d})^{-1}(D_\delta - x') + x' \subset F_\delta$$

where $\lambda = \text{vol}_d(D_\delta)^{-1}$ and x' is the centroid of D_δ .

PROOF. Let δ_0 be such that $\text{vol}_d(D_{\delta_0}) > 8^d$. We use the notation $(D_\delta)_\lambda$ for the convex floating body with parameter $\lambda > 0$ corresponding to the uniform probability measure on D_δ . If \mathfrak{H} is any half-space with $\mu(\mathfrak{H}) < \delta$, then $\text{vol}_d(\mathfrak{H} \cap D_\delta) < 1$. Hence $(D_\delta)_\lambda \subset F_\delta$, where $\lambda = \text{vol}_d(D_\delta)^{-1}$. The result now follows from inequality (3.6). \square

Lemma 10. *Let $K, L \subset \mathbb{R}^d$ be convex bodies such that there exist $x, x' \in \text{int}(K \cap L)$ and $0 < r < (8d)^{-1}$ for which*

$$(7.6) \quad (1+r)^{-1}(K-x) + x \subset L \subset (1+r)(K-x') + x'$$

Then

$$(7.7) \quad d_{\mathfrak{L}}(K, L) \leq 1 + 8dr$$

PROOF. Since the statement of the theorem is invariant under affine transformations of K and L , we may assume without loss of generality that the John ellipsoid of K is B_2^d . Hence $B_2^d \subset K \subset dB_2^d$ and $\|x\|_2, \|x'\|_2 \leq d$. Note also that $L \subset 3dB_2^d$. Using these facts and manipulating (7.6) in the obvious way, we see that both of the following relations hold

$$\begin{aligned} L &\subset K + 2drB_2^d \\ K &\subset L + 4drB_2^d \end{aligned}$$

By definition of the Hausdorff distance, $d_{\mathcal{H}}(K, L) \leq 4dr$. Since $B_2^d \subset K$, $d_{\mathfrak{L}}(K, L) \leq (1 - 4dr)^{-1} \leq 1 + 8dr$. \square

PROOF OF EQUATION (2.4). Since $\lim_{\delta \rightarrow 0} p(\delta) = \lim_{\delta \rightarrow 0} \lambda(\delta) = 0$, equation (2.4) now follows from (7.3), (7.5) and (7.7). \square

Remark 1. *There is no lower bound on the growth rate of $\text{vol}_d(D_\delta)$, indeed the function could grow arbitrarily slowly. However in the case of the Schechtman-Zinn distributions, $\text{vol}_d(D_\delta) = (\log(c_p^d/\delta))^{d/p} \text{vol}_d(B_p^d)$ and we leave it to the reader to combine this with (7.3), (7.2) and (7.5) to obtain a quantitative upper bound on $d_{\mathfrak{L}}(F_\delta, D_\delta)$.*

PROOF OF EQUATION (2.5). Let $\varepsilon > 0$ be given. Using the notation from the proof of theorem 1, for any $\theta \in S^{d-1}$ we define

$$f_\theta(t) = -\frac{d}{dt} \mu(\mathfrak{H}_{\theta,t})$$

This function is the density of a log-concave probability measure on \mathbb{R} with cumulative distribution function $J_\theta(t) = 1 - \mu(\mathfrak{H}_{\theta,t})$. By Fubini's theorem we have

$$f_\theta(t) = Rf(\mathcal{H}_{\theta,t})$$

Define $\alpha = \inf\{f_\theta(0) : \theta \in S^{d-1}\}$. By (3.8) there exists $t_0 > 0$ such that if $\beta = \sup\{f_\theta(t_0) : \theta \in S^{d-1}\}$, then $\beta < \alpha$. Since f is non-vanishing, S^{d-1} is compact and the function $\theta \mapsto f_\theta(t)$ is continuous, $\beta > 0$. Define $g_\theta(t) = -\log f_\theta(t)$ and let $\lambda = t_0^{-1}(\log \alpha - \log \beta)$ and $\Delta = \max\{1, \lambda^{-1} \log \lambda^{-1}\}$. By definition of α, β and λ , for all $\theta \in S^{d-1}$ we have $t_0^{-1}(g_\theta(t_0) - g_\theta(0)) \geq \lambda$. By convexity of g_θ , if $u > v \geq t_0$ then $g_\theta(u) \geq g_\theta(v) + \lambda(u - v)$, which translates into $f_\theta(u) \leq f_\theta(v)e^{-\lambda(u-v)}$. Let $\delta_0 < \inf\{f_\theta(t_0 + 1) : \theta \in S^{d-1}\}$ be such that $\Delta\varepsilon^{-1}B_2^d \subset F_{\delta_0}$. Consider any $\delta < \delta_0$ and momentarily fix $\theta \in S^{d-1}$. Let $s = J_\theta^{-1}(\delta)$ and denote by $t = f_\theta^{-1}(\delta)$ the

unique positive number such that $f_\theta(t) = \delta$. Consider the hyperplane $\mathcal{H}_{\theta,t}$ and the half-space $\mathfrak{H}_{\theta,s}$. Note that

$$\mu(\mathfrak{H}_{\theta,s}) = Rf(\mathcal{H}_{\theta,t}) = \delta$$

By log-concavity we have $f_\theta(u) \geq \delta_0$ for all $0 < u < t_0 + 1$, hence $t > t_0 + 1$. By the fundamental theorem of calculus and the fact that $f_\theta(u) \geq \delta$ for all $u \in [t-1, t]$ we have

$$\begin{aligned} \mu(\mathfrak{H}_{\theta,t-1}) &> \mu\{x \in \mathbb{R}^d : t-1 \leq \langle \theta, x \rangle \leq t\} \\ &= \int_{t-1}^t f_\theta(u) du \\ &\geq \delta \end{aligned}$$

hence $\mathfrak{H}_{\theta,s} \subset \mathfrak{H}_{\theta,t-1}$ which implies that $s > t-1 > t_0$. Thus, if $s \leq t$ then $|s-t| \leq 1$. If $s > t$ then

$$\begin{aligned} \delta &= \int_s^\infty f_\theta(u) du \\ &\leq f_\theta(s) \int_s^\infty e^{-\lambda(u-s)} du \\ &\leq \delta e^{-\lambda(s-t)} \lambda^{-1} \end{aligned}$$

from which it follows that $s-t \leq \lambda^{-1} \log \lambda^{-1}$. Either way, $|s-t| \leq \max\{1, \lambda^{-1} \log \lambda^{-1}\} = \Delta$. Since $\Delta \varepsilon^{-1} B_2^d \subset F_{\delta_0}$, it follows that $(1-\varepsilon)s \leq t \leq (1+\varepsilon)s$. Since this holds for all $\theta \in S^{d-1}$ we have

$$(1-\varepsilon)F_\delta \leq R_\delta \leq (1+\varepsilon)F_\delta$$

□

8. Proof of theorem 4

If Ω is a convex subset of a real vector space and \mathcal{K}_d is the collection of all convex bodies in \mathbb{R}^d , then we define a function $\kappa : \Omega \rightarrow \mathcal{K}_d$ to be concave if for all $x, y \in \Omega$ and all $\lambda \in (0, 1)$ we have

$$\lambda \kappa(x) + (1-\lambda) \kappa(y) \subset \kappa(\lambda x + (1-\lambda)y)$$

If Ω has an ordering then we define κ to be non-decreasing if for all $x, y \in \Omega$ with $x \leq y$ we have $\kappa(x) \subset \kappa(y)$.

Lemma 11. *If $\kappa : [0, \infty) \rightarrow \mathcal{K}_d$ is concave, non-decreasing and $\cup_{t \in [0, \infty)} \kappa(t) = \mathbb{R}^d$, then the function $g : \mathbb{R}^d \rightarrow [0, \infty)$ defined by*

$$(8.1) \quad g(x) = \inf\{t \geq 0 : x \in \kappa(t)\}$$

is convex. Furthermore, κ is continuous with respect to the Hausdorff distance and for all $t > 0$

$$(8.2) \quad \kappa(t) = \{x \in \mathbb{R}^d : g(x) \leq t\}$$

PROOF. By translation we may assume that $0 \in \kappa(0)$. For any $0 < \varepsilon < t$ we have the convex combination

$$t = \frac{\varepsilon}{t+\varepsilon} 0 + \frac{t}{t+\varepsilon} (t+\varepsilon)$$

Exploiting the concavity of κ , this leads to

$$\kappa(t + \varepsilon) \subset \frac{t + \varepsilon}{t} \kappa(t)$$

Similarly,

$$\frac{t - \varepsilon}{t} \kappa(t) \subset \kappa(t - \varepsilon)$$

Hence κ is continuous with respect to the Hausdorff distance. By definition of g , $\kappa(t) \subset \{x \in \mathbb{R}^d : g(x) \leq t\}$. Since $\kappa(t)$ is a closed set, if $x \notin \kappa(t)$ then $d(x, \kappa(t)) > 0$ and by continuity of κ , $g(x) > t$. This implies (8.2). Consider any $x, y \in \mathbb{R}^d$ and $\lambda \in (0, 1)$. Let $t = g(x)$ and $s = g(y)$. By (8.2), $x \in \kappa(t)$ and $y \in \kappa(s)$. Therefore

$$\begin{aligned} \lambda x + (1 - \lambda)y &\in \lambda \kappa(t) + (1 - \lambda) \kappa(s) \\ &\subset \kappa(\lambda t + (1 - \lambda)s) \end{aligned}$$

This implies that $g(\lambda x + (1 - \lambda)y) \leq \lambda t + (1 - \lambda)s$ which shows that g is convex. \square

Note that the function g is a generalization of the Minkowski functional of a convex body K , in which case $\kappa(t) = tK$. Including $\{0\}$ as an honorary member of \mathcal{K}_d does no harm to the preceding lemma. If $(K_n)_{n=1}^\infty$ is a sequence of convex bodies then we define the corresponding Minkowski series as,

$$\sum_{n=1}^\infty K_n = \left\{ \sum_{n=1}^\infty x_n : \forall n, x_n \in K_n \right\}$$

where we take $\sum x_n$ to have meaning only if it converges. We leave the easy proof of the following lemma to the reader.

Lemma 12. *For each $n \in \mathbb{N}$, let $\alpha_n : [0, \infty) \rightarrow [0, \infty)$ be a concave function and let K_n be a convex body with $0 \in K_n$. Provided that*

$$\sum_{n=1}^\infty \alpha_n(t) \text{diam}(K_n) < \infty$$

for all $t \geq 0$, then the function $\kappa : [0, \infty) \rightarrow \mathcal{K}_d$ defined by

$$\kappa(t) = \sum_{n=1}^\infty \alpha_n(t) K_n$$

is concave.

The space \mathcal{K}_d is separable in the Banach-Mazur distance and we shall use a sequence $(K_n)_{n=1}^\infty$ that is Banach-Mazur dense in \mathcal{K}_d . Since the Banach-Mazur distance is blind to affine transformations we can assume that the John ellipsoid of each K_n is B_2^d . As coefficients, we shall use the functions

$$\alpha_n(t) = \begin{cases} 2^{-n^2} t & : 0 \leq t \leq 2^{2n^2} \\ 2^{n^2} & : 2^{2n^2} < t < \infty \end{cases}$$

Note that for large values of n , the dominant coefficient at the value $t = 2^{2n^2}$ is α_n . In fact $\sum_{j \neq n} \alpha_j(2^{2n^2})$ is much smaller than $\alpha_n(2^{2n^2})$,

$$\begin{aligned} \sum_{j \neq n} \alpha_j(2^{2n^2}) &= \sum_{j=1}^{n-1} 2^{j^2} + 2^{2n^2} \sum_{j=n+1}^{\infty} 2^{-j^2} \\ &\leq \sum_{j=1}^{n-1} 2^{nj} + 2^{2n^2} \sum_{j=n+1}^{\infty} 2^{-nj} \\ &\leq 2^{n^2-n+2} \\ &= 2^{-n+2} \alpha_n(2^{2n^2}) \end{aligned}$$

Hence,

$$d_{BM}(\kappa(2^{2n^2}), K_n) \leq 1 + 2^{-n+2}d$$

Thus the sequence $(\kappa(n))_{n=1}^{\infty}$ is dense in \mathcal{K}_d . Since each coefficient α_n is non-decreasing and concave, κ is concave and the function g as defined by (8.1) is convex. Clearly, $\lim_{x \rightarrow \infty} g(x) = \infty$. For some $c > 0$, the function

$$f(x) = 2^{-g(cx)}$$

is the density of a log-concave probability measure μ on \mathbb{R}^d . For each $n \in \mathbb{N}$, $D_{2^{-n}} = \{x \in \mathbb{R}^d : f(x) \geq 2^{-n}\} = \{x \in \mathbb{R}^d : g(cx) \leq n\} = c^{-1}\kappa(n)$. Hence the sequence $(D_{1/n})_{n=1}^{\infty}$ is dense in \mathcal{K}_d . By (2.4), the sequence $(F_{1/n})_{n=3}^{\infty}$ is also dense in \mathcal{K}_d .

We now use theorem 1 with $q = 1$. Let $\tilde{\mathcal{K}}_d$ denote a countably dense subset of \mathcal{K}_d and let $K \in \tilde{\mathcal{K}}_d$. Note that there exists an increasing sequence of natural numbers $(k_n)_1^{\infty}$ such that $\lim_{n \rightarrow \infty} d_{BM}(F_{1/k_n}, K) = 1$ and $\sum_{n=1}^{\infty} \tilde{c}(\log k_n)^{-1} < \varepsilon$. By (2.2),

$$\lim_{n \rightarrow \infty} d_{BM}(P_{k_n}, K) = 1$$

with probability at least $1 - \varepsilon$. Since this holds for all $\varepsilon > 0$, $K \in cl_{BM}\{P_n : n \in \mathbb{N}, n \geq d+1\}$ almost surely, where cl_{BM} denotes closure in \mathcal{K}_d with respect to d_{BM} . Since this holds for all $K \in \tilde{\mathcal{K}}_d$ and $\tilde{\mathcal{K}}_d$ is countable, $\tilde{\mathcal{K}}_d \subset cl_{BM}\{P_n : n \in \mathbb{N}, n \geq d+1\}$ almost surely. The result now follows since $\tilde{\mathcal{K}}_d$ is dense in \mathcal{K}_d .

References

- [1] Ball, K.: An elementary introduction to modern convex geometry. Flavors of Geometry, MSRI Publications 31 (1997)
- [2] Ball, K.: Ellipsoids of maximal volume in convex bodies. *Geom. Dedicata* **41** (2), 241-250 (1992)
- [3] Bárány, I., Larman, D. G.: Convex bodies, economic cap coverings, random polytopes. *Mathematika* **35** (2), 274-291 (1988)
- [4] Bárány, I., Vu, V.: Central limit theorems for Gaussian polytopes. *Ann. Probab.* **35** (4), 1593-1621 (2007)
- [5] Bobkov, S.: On concentration of distributions of random weighted sums. *Ann. Probab.* **31** (1), 195-215 (2003)
- [6] Fresen, D.: Sensitivity of the floating body. Unpublished manuscript.
- [7] Gnedenko, B.: Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. Math.* **44**, 423-453 (1943)
- [8] Gruber, P. M.: Aspects of approximation of convex bodies. *Handbook of Convex Geometry*, Vol. A. 319-345, North Holland, Amsterdam (1993)
- [9] Klartag, B., Milman, V.: Geometry of log-concave functions and measures. *Geom. Dedicata* **112**, 169-182 (2005).

- [10] Koldobsky, A.: Fourier Analysis in Convex Geometry, Mathematical Surveys and Monographs, American Mathematical Society, Providence RI 2005.
- [11] Lovász, L., Vempala, S.: The geometry of logconcave functions and sampling algorithms. Random Structures Algorithms **30** (3), 307-358 (2007)
- [12] Matoušek, J.: Lectures on Discrete Geometry. Graduate Texts in Mathematics. Springer-Verlag, New York (2002)
- [13] Milman, V., Schechtman, G.: Asymptotic Theory of Finite-Dimensional Normed Spaces. Lect. Notes Math., vol. 1200. Springer, Berlin (1986)
- [14] Naor, A., Romik, D.: Projecting the surface measure of the sphere of ℓ_p^n . Ann. Inst. H. Poincaré Probab. Statist. **39** (2), 241-261 (2003)
- [15] Raynaud, H.: Sur l'enveloppe convexe des nauges de points aléatoires dans \mathbb{R}^n . J. Appl. Probab., **7**, 35-48 (1970)
- [16] Schechtman, G., Zinn, J.: On the volume of the intersection of two L_p^n balls. Proc. Amer. Math. Soc. **110** (1), 217-224 (1990)
- [17] Schütt, C., Werner, E.: The convex floating body. Math. Scand. **66**, 275-290 (1990)

UNIVERSITY OF MISSOURI

E-mail address: `djfb6b@mail.missouri.edu`